MTH 166

Lecture-16

Method of Undeterminant Coefficients

Topic:

Solution of Non-Homogeneous LDE with Constant coefficients.

Learning Outcomes:

To use Method of Undeterminant coefficients to solve Non-Homogeneous LDE with constant coefficients.

Method of Undeterminant Coefficients:

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$$ay'' + by' + cy = r(x) \qquad (1)$$

In this method, we guess the trial solution (P.I.) y_p by looking at the type of $r(x)$.

As y_p is solution of equation (1), so it satisfies (1) from where we calculate the Undeterminant coefficients by comparing like coefficients on both sides.

For example:

1. If $r(x) = e^{\alpha x}$, then assumed trial solution: $y_p = A\, e^{\alpha x}$

2. If $r(x) = x^2$, then assumed trial solution: $y_p = A\, x^2 + Bx + C$

3. If $r(x) = \cos ax$, then assumed trial solution: $y_p = A \cos x + B \sin x$

where A, B, C are Undeterminant coefficients to be evaluated.

Problem 1. Find the general solution of: $4y'' - y = e^{3x}$

Solution: The given equation is:

$$4y'' - y = e^{3x} \qquad (1)$$

S.F. : $(4D^2 - 1)y = e^{3x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (4D^2 - 1)$ and $r(x) = e^{3x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (4D^2 - 1) = 0 \quad \Rightarrow D^2 = \frac{1}{4}$

$\Rightarrow D = \frac{1}{2}, -\frac{1}{2}$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$$\Rightarrow y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$$

To find Particular Integral (P.I.):

Since $r(x) = e^{3x}$, So, let trial solution be: $y_p = ae^{3x}$

$\Rightarrow y_p{}' = 3ae^{3x}$ $\Rightarrow y_p{}'' = 9ae^{3x}$

Since y_p is a solution of equation (1)

So, $4y_p{}'' - y_p = e^{3x}$

$\Rightarrow 4(9ae^{3x}) - ae^{3x} = e^{3x}$ $\Rightarrow 35ae^{3x} = e^{3x}$ $\Rightarrow 35a = 1$ $\Rightarrow a = \frac{1}{35}$

$\therefore y_p = \frac{1}{35}e^{3x}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = \left(c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}\right) + \frac{1}{35}e^{3x}$ **Answer.**

Problem 2. Find the general solution of: $y'' - 3y' - 10y = x^2 + 1$

Solution: The given equation is:

$$y'' - 3y' - 10y = x^2 + 1 \qquad (1)$$

S.F. : $(D^2 - 3D - 10)y = x^2 + 1$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 3D - 10)$ and $r(x) = x^2 + 1$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 3D - 10) = 0 \quad \Rightarrow (D - 5)(D + 2) = 0$

$\Rightarrow D = 5, -2$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 e^{5x} + c_2 e^{-2x}$

To find Particular Integral (P.I.):

Since $r(x) = x^2 + 1$, So, let trial solution be: $y_p = ax^2 + bx + c$

$\Rightarrow {y_p}' = 2ax + b \qquad \Rightarrow {y_p}'' = 2a$

Since y_p is a solution of equation (1)

So, ${y_p}'' - 3{y_p}' - 10y_p = x^2 + 1$

$\Rightarrow 2a - 3(2ax + b) - 10(ax^2 + bx + c) = x^2 + 1$

$\Rightarrow -10ax^2 - (6a + 10b)x + (2a - 3b - 10c) = x^2 + 1$

Comparing the like coefficients:

Coeff. of x^2: $-10a = 1 \quad \Rightarrow a = \frac{1}{10}$

Coeff. of x : $-(6a + 10b) = 0 \quad \Rightarrow 10b = -6a \quad \Rightarrow b = -\frac{6}{100}$

Constant terms: $(2a - 3b - 10c) = 1 \quad \Rightarrow 10c = 2a - 3b - 1$

$$\Rightarrow c = \frac{1}{10}(2a - 3b - 1) = \frac{1}{10}\left(2\left(\frac{1}{10}\right) - 3\left(\frac{-6}{100}\right) - 1\right) = -\frac{62}{1000}$$

Putting back values of a, b, c in $y_p = ax^2 + bx + c$

$$y_p = \frac{1}{10}x^2 - \frac{6}{100}x - \frac{62}{1000}$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 e^{5x} + c_2 e^{-2x}) + \frac{1}{10}x^2 - \frac{6}{100}x - \frac{62}{1000}$$ **Answer.**

Problem 3. Find the general solution of: $y'' + y' - 6y = 39\cos 3x$

Solution: The given equation is:

$$y'' + y' - 6y = 39\cos 3x \qquad (1)$$

S.F. : $(D^2 + D - 6)y = 39\cos 3x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + D - 6)$ and $r(x) = 39\cos 3x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + D - 6) = 0 \quad \Rightarrow (D - 2)(D + 3) = 0$

$\Rightarrow D = 2, -3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-3x}$

To find Particular Integral (P.I.):

Since $r(x) = 39\cos 3x$, So, let trial solution be: $y_p = a\cos 3x + b\sin 3x$

$\Rightarrow y_p{}' = -3a\sin 3x + 3b\cos 3x$ $\Rightarrow y_p{}'' = -9a\cos 3x - 9b\sin 3x$

Since y_p is a solution of equation (1)

So, $y_p{}'' + y_p{}' - 6y_p = 39\cos 3x$

$\Rightarrow (-9a\cos 3x - 9b\sin 3x) + (-3a\sin 3x + 3b\cos 3x) - 6(a\cos 3x + b\sin 3x) = 39\cos 3x$

$\Rightarrow (-15a + 3b)\cos 3x + (-3a - 15b)\sin 3x = 39\cos 3x + 0\sin 3x$

Comparing the like coefficients:

Coeff. of $\cos 3x$: $-15a + 3b = 39$ (2)

Coeff. of $\sin 3x$: $-3a - 15b = 0$ (3)

Solving equations (2) and (3):

$-15a + 3b = 39$ (2) $\times 3$

$-3a - 15b = 0$ (3) $\times 15$ and subtracting, we get:

$234b = 127 \Rightarrow b = \frac{1}{2}$

Put value of b in equation (3): $3a = -15b = -\frac{15}{2} \quad \Rightarrow a = -\frac{5}{2}$

Putting back values of a, b in $y_p = a \cos 3x + b \sin 3x$

$$y_p = -\frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 e^{2x} + c_2 e^{-3x}) - \frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$$ **Answer.**